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COMPLEX MODULI FOR SEISMIC RESPONSE ANALYSIS OF GROUND

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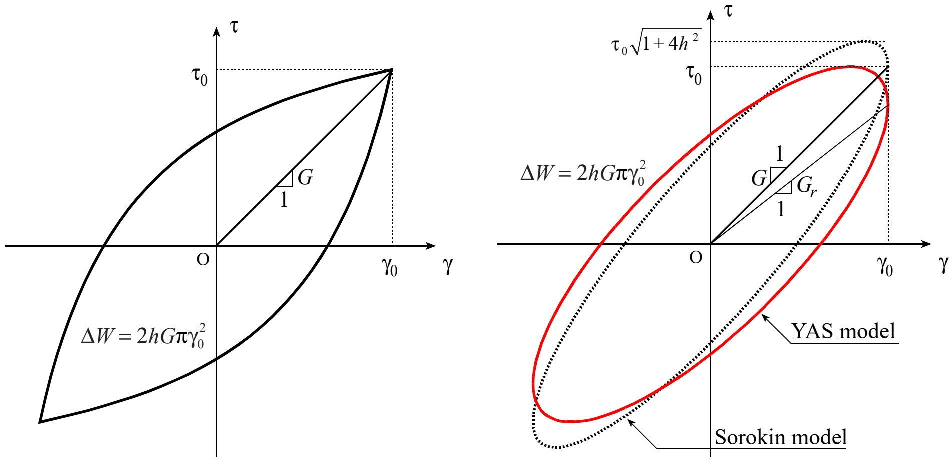
ABSTRACT: Frequency independent stress–strain relationships for the complex moduli used in a seismic response analysis of ground are discussed. In addition, a new complex modulus for the seismic response analysis of ground, named YAS (Yoshida–Adachi–Sorokin) model, is proposed. This model is designed so that the maximum stress and the hysteretic absorption energy agree with those of the cyclic shear test. It is shown that the complex modulus, in which the maximum stress and the damping ratio agree with the test result, is possible only when the damping ratio is less than 0.5. Then, three complex moduli, the Sorokin model used in the original SHAKE, the Lysmer model proposed to improve Sorokin model and used widely in the equivalent linear method, and the YAS model, are compared and discussed. The obtained conclusions are as follows. The Sorokin model overestimates the maximum shear stress. The Lysmer model gives the same maximum stress as the cyclic shear test result, but it underestimates the hysteretic absorption energy. Underestimation of the energy absorption is less than 5% for damping ratio less than 0.3 which is the maximum important damping ratio in practical use.

Keywords: *Complex modulus, Seismic response analysis, Equivalent linear method*

1. INTRODUCTION

The computer program SHAKE[1)] developed by Schnabel et al. is still widely used in the seismic response analysis of ground. In SHAKE, stress–strain relationships that have hysteretic characteristics are linearized by introducing a complex modulus and solving the equation of motion in the frequency domain. A separation of variables method is applied to the governing equation on ground displacements expressed as a function of time and space, and the solutions in each frequency are summed up to obtain the solution under nonstationary earthquake motion input.

Stress–strain relationships are represented by a complex number in the complex modulus method. This model absorbs energy caused by the hysteretic behavior (hysteretic absorption energy, hereafter) by the phase difference between the real and imaginary parts; the absorbed energy is set equal to the absorbed energy by the nonlinear behavior of soil. Lysmer, one of the developers of SHAKE, proposed a new



(a) Cyclic shear test (b) Complex modulus

Fig. 1 Comparison of stress–strain relationships

complex modulus from the analysis of a single-degree-of-freedom (SDOF) system subjected to a sinusoidal loading[2]. He showed that the displacements by using the complex modulus of SHAKE and by using the velocity proportional damping do not agree, and the proposed complex modulus gives the same displacements. Since Lysmer proposed to replace the complex modulus in SHAKE with the complex modulus he proposed, many computer programs such as SHAKE91[3], the two-dimensional computer program FLUSH[4], the improved equivalent linear method DYNEQ[5], etc. use this complex modulus.

Lysmer's discussion is based on the SDOF system, but the mechanical properties as a complex modulus to be used in the analysis of the seismic response analysis of ground is not discussed. In this paper, we propose a new complex modulus named YAS (Yoshida–Adachi–Sorokin) model. Then mechanical properties of the Sorokin model (complex modulus in the original SHAKE), the Lysmer model (complex modulus proposed by Lysmer), and the YAS model are made clear.

2. SHAKE AND SOROKIN MODEL

Expressions of the real numbers and the complex numbers are distinguished in this paper; a complex number is expressed by putting an overline. For example, let $\bar{\gamma}$ denote a complex number, then γ is the real part of $\bar{\gamma}$, i.e., $\gamma = \mathrm{Re}(\bar{\gamma})$.

Although it is not written in the manual of SHAKE[1], the complex modulus used in SHAKE is the Sorokin model, which is the complex expression of the Voigt model. Therefore, the Sorokin model is explained at first, then the relationship with SHAKE is discussed.

2.1 Complex modulus by the Sorokin model

The shape of the stress–strain relationships obtained by the cyclic shear test is a spindle shape as typically shown in Fig. 1(a). It is characterized by the secant shear modulus G and the hysteretic absorption energy ΔW. They change depending on the strain amplitude γ_0, but they are supposed not to depend on the loading frequency. Among them, the hysteretic absorption energy is usually represented as the damping ratio h, which is a dimensionless expression of the absorbed energy defined as the ratio of ΔW to the strain energy W, i.e.

$$h = \frac{1}{4\pi}\frac{\Delta W}{W}, \quad W = \frac{1}{2}G\gamma_0^2 \tag{1a, b}$$

Thus, the result of the cyclic shear test is expressed as $G-\gamma_0$ and $h-\gamma_0$ relationships. In practical use, however, since the shear strain is usually expressed as γ, they are usually expressed as $G-\gamma$ and $h-\gamma$ relationships.

The Voigt model, a mechanical model by connecting a spring and a dashpot in parallel, is employed to express the stress–strain relationships such as Fig. 1(a). The stress–strain relationship is expressed as

$$\tau = G\gamma + C\dot{\gamma} \tag{2}$$

where C denotes damping coefficient, τ and γ denote stress and strain, respectively, and dot denotes time derivative.

We consider the steady-state vibration (harmonic vibration), where the shear strain is defined as

$$\gamma = \gamma_0 \cos\omega t \quad \text{or} \quad \cos\omega t = \gamma / \gamma_0 \tag{3}$$

where ω denotes circular frequency. Substitution of this equation into Eq. (2) and using the well-known relationship $\sin\omega t = \pm(1-\cos^2 \omega t)^{0.5}$, we obtain

$$\tau = G\gamma_0 \cos\omega t - C\gamma_0 \omega \sin\omega t = G\gamma \pm C\omega\sqrt{\gamma_0^2 - \gamma^2} \tag{4}$$

Equation (4) is the stress–strain relationships of the Voigt model. Since this equation includes circular frequency ω, the stress–strain relationships in Eq. (4) are frequency dependent. Next, we consider that the dashpot has frequency-dependent nature and define a new parameter β as

$$\omega C = 2\beta G \quad \text{or} \quad C = 2\beta G / \omega \tag{5}$$

Then the frequency-independent stress–strain relationships are obtained from Eq. (4) as

$$\tau = G\gamma \pm 2\beta G\sqrt{\gamma_0^2 - \gamma^2} \tag{6}$$

The first term of this model is an elastic relation that comes from the spring of the Voigt model, and the second term becomes a shape of the ellipse that comes from the dashpot. The double sign corresponds to the upper and lower half of the ellipse.

The hysteretic absorption energy ΔW_V, i.e., the area of the ellipse of the stress–strain relationships in Eq. (6) becomes

$$\Delta W_V = \int_{-\gamma_0}^{\gamma_0} \left\{ \left(G\gamma + 2\beta G\sqrt{\gamma_0^2 - \gamma^2} \right) - \left(G\gamma - 2\beta G\sqrt{\gamma_0^2 - \gamma^2} \right) \right\} d\gamma = 2\beta G\pi\gamma_0^2 \tag{7}$$

Here, the subscript V of ΔW_V indicates the Voigt model. Assuming that ΔW_V is equal to the hysteretic absorption energy ΔW, we obtain

$$\beta = h \tag{8}$$

This means that parameter β becomes equal to the damping constant h. However, this relation does not always hold in the models shown below. Therefore, we distinguish between the material damping constant β defined by Eq. (5) and the damping ratio h defined in Eq. (1a). This is discussed in section 6.1 in detail.

Sorokin gives the strain in Eq. (3) as a complex number in Eq. (9) and expresses the stress–strain relationships as given in Eq. (10)[6].

$$\overline{\gamma} = \gamma_0 e^{i\omega t} \tag{9}$$

$$\overline{\tau} = G\overline{\gamma} + C\dot{\overline{\gamma}} \tag{10}$$

The following stress–strain relationships are obtained by substituting Eq. (9) into Eq. (10) and by using the relationships in Eq. (5),

$$\overline{\tau} = G(1+2i\beta)\gamma_0 e^{i\omega t} = \overline{G}_S^*\overline{\gamma} \tag{11}$$

where

$$\overline{G}_S^* = G(1+2i\beta) \tag{12}$$

is called a complex modulus[1]. The asterisk * is not necessary for the definition because the overline, which indicates a complex number, is put on G_S. However, since * is used in many technical papers, it is also used in this paper for convenience. The stress–strain relationships that have hysteretic damping can be expressed as a linear equation in Eq. (11). The Voigt model is used in order to drive Eq. (11), but this equation can be obtained from the Maxwell model, a model connecting a spring and a dashpot in series, by assuming that h^2 is sufficiently small compared with unity and can be neglected[7].

The frequency-independent stiffness obtained by introducing the complex stresses and the strains is called the Sorokin's hypothesis or Sorokin's damping (e.g., Konishi et al.[8] and Hricko[9]). Thus this complex modulus is called the Sorokin model in this paper; the subscript S in Eqs. (11) and (12) indicates that this complex modulus is the Sorokin model.

The following stress–strain relationships are obtained by retrieving the real part of Eq. (11)

$$\begin{aligned} &\tau = G\gamma_0(\cos\omega t - 2\beta\sin\omega t) = G\gamma_0\sqrt{1+4\beta^2}\cos(\omega t+\phi) \\ &\tan\phi = 2\beta \end{aligned} \tag{13a, b}$$

This equation is equivalent to Eq. (6), although the expressions are different.

The hysteretic absorption energy ΔW_S under the shear strain amplitude γ_0 is evaluated by the same procedure with Eq. (7) as

$$\Delta W_S = 2G\beta\pi\gamma_0^2 \tag{14}$$

Then, ΔW_S is equal to ΔW_V in Eq. (7), and thus is equal to the hysteretic absorption energy in Fig. 1(a).

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1 The damping ratio h is frequently used instead of the material damping constant β in Eq. (12) in many textbooks and technical papers. This is possible because Eq. (8) holds for the Voigt model and the Sorokin model. In the models shown later, however, β and h are strictly distinguished; they are defined in Eqs. (5) and (1a), respectively, and are different quantities.

2.2 Method by SHAKE

The computer program SHAKE[1)] deals with the one-dimensional ground. The equation of motion is written as

$$\rho \frac{\partial^2 \overline{u}}{\partial t^2} = G \frac{\partial^2 \overline{u}}{\partial z^2} + C \frac{\partial^3 \overline{u}}{\partial t \partial z^2} \tag{15}$$

where ρ denotes density and C denotes damping coefficient. Displacement $\overline{u}$ is a function with respect to both time and space; therefore, it is written as $\overline{u} = \overline{u}(z,t)$ where z denotes the coordinate axis directing vertical downward. In order to solve this partial differential equation, the variable separation method is employed such as

$$\overline{u}(z,t) = \overline{U}(z) e^{i\omega t} \tag{16}$$

Then, Eq. (15) yields[2)]

$$\left(G + i\omega C\right) \frac{d^2 \overline{U}}{dz^2} = -\rho \omega^2 \overline{U}(z) \tag{17}$$

The frequency-independent equation is obtained by using Eq. (5) and by substituting the complex modulus $\overline{G}_S^*$ in Eq. (12).

Here, Eq. (15) is derived by the following procedure. The stress–strain relationship is shown in Eq (2). Substitution of Eq. (5) and the strain–displacement relationship $\gamma = \partial u / \partial z$ into this equation, we obtain

$$\tau = G \frac{\partial u}{\partial z} + \frac{2\beta G}{\omega} \frac{\partial^2 u}{\partial t \partial z} \tag{18}$$

On the other hand, the equation of motion in the horizontal direction of the one-dimensional ground is as follows,

$$\frac{\partial \tau}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} \tag{19}$$

Equation (15) is obtained by substituting Eq. (18) after partial differentiation with respect to z into Eq. (19). Thus, it becomes clear that SHAKE uses the Sorokin model.

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[2] The minus sign on the right side of Eq. (17) is not shown in the manual of SHAKE[1)], but it is not correct. However, the final equation is correct because their second mistake cancels this error; they set the square of the imaginary unit equal to 1 instead of −1.

3. YAS MODEL

The maximum shear stress under the cyclic shear test with shear strain amplitude γ_0 is $\tau_0 = G\gamma_0$. On the other hand, the maximum shear stress τ_{max} of the Sorokin model is calculated from the stress–strain relationships in Eq. (13a), which result in

$$\tau_{max} = G\gamma_0\sqrt{1+4h^2} = \tau_0\sqrt{1+4h^2} \tag{20}$$

The maximum shear stress is $\sqrt{1+4h^2}$ times larger than that of the laboratory test. It is noted that β is used in Eq. (13a), but h is used in this equation by using the relation in Eq. (8).

A larger evaluation of the maximum shear stress leads to a larger evaluation of the maximum acceleration (e.g., Yoshida[10]). Therefore, a new complex modulus that results in the maximum shear stress that is the same as that of the laboratory test $\tau_0 = G\gamma_0$ is examined.

The stiffness of the elastic part (proportional to strain) of the Sorokin model is the same as the secant modulus G obtained in the laboratory test, but, as shown in Fig. 1(b), the elastic stiffness becomes smaller than G when the maximum shear stress is set as τ_0. This reduced stiffness is denoted as G_r, then Eq. (12) is re-written as

$$\overline{G}_Y^* = G_r + 2ihG \tag{21}$$

The subscript Y in the complex modulus is used to distinguish it from the other models.

As can be seen from Eq. (7), the hysteretic absorption energy resulted from the stress–strain relationships of the complex modulus are controlled only by β. Since the YAS model and the Sorokin model use the same imaginary part $h = \beta$, the hysteretic absorption energy ΔW_Y is equal to ΔW_S, and is expressed as

$$\Delta W_Y = \Delta W_S = 2\pi G h \gamma_0^2 \tag{22}$$

Naturally, this hysteretic absorption energy is equal to that of the laboratory test ΔW.

The complex stress $\overline{\tau}$ under the complex strain $\overline{\gamma}$ becomes

$$\overline{\tau} = G_Y^*\overline{\gamma} = (G_r + 2iGh)\gamma_0 e^{i\omega t} \tag{23}$$

The stress–strain relationships by the YAS model are calculated as the real part of this equation, resulting in

$$\begin{aligned} \tau &= \mathrm{Re}(\overline{\tau}) = \gamma_0\left(G_r\cos\omega t - 2Gh\sin\omega t\right) = \gamma_0\sqrt{G_r^2 + 4G^2h^2}\cos(\omega t + \phi) \\ \tan\phi &= 2Gh/G_r \end{aligned} \tag{24a, b}$$

In order to set the maximum shear stress $\tau_0 = G\gamma_0$ in this equation, it is necessary that $G = \sqrt{G_r^2 + 4G^2h^2}$. Then the real part G_r of the YAS model yields

$$G_r = \sqrt{1-4h^2}\,G \tag{25}$$

The final form of the YAS model is obtained by substituting this equation into Eq. (21) as

$$\bar{G}_Y^* = G\left(\sqrt{1-4h^2} + 2ih\right) \tag{26}$$

Then the stress–strain relationships yield

$$\begin{aligned} \tau &= G\gamma_0\left(\sqrt{1-4h^2}\cos\omega t - 2h\sin\omega t\right) = G\gamma_0\cos(\omega t+\phi) \\ \tan\phi &= \frac{2h}{\sqrt{1-4h^2}} \end{aligned} \tag{27a, b}$$

Both the maximum stress and the hysteretic absorption energy (or damping ratio) in the YAS model agree with those of the laboratory test.

Stress–strain relationships obtained by this model are shown in Fig. 2 with h as parameters. The real part of the complex modulus becomes 0 at $h = 0.5$ (50%), at which the hysteresis curve becomes a complete circle. This indicates that the complex modulus that gives the same maximum shear stress and hysteretic absorption energy as those in the laboratory test is possible up to $h = 0.5$.

4. LYSMER MODEL

Lysmer[2)] proposed a new complex modulus by focusing on the SDOF system subjected to harmonic excitation. This model is called Lysmer model in this paper. In this chapter, behaviors of the SDOF system are compared following the proposal by Lysmer, and their mechanical properties are discussed.

4.1 Response of SDOF system and Lysmer model

The behavior of the SDOF system that Lysmer discussed is first examined following the procedure by Lysmer. The equation of motion of the SDOF system is expressed by using mass m, spring constant k, and damping coefficient C as

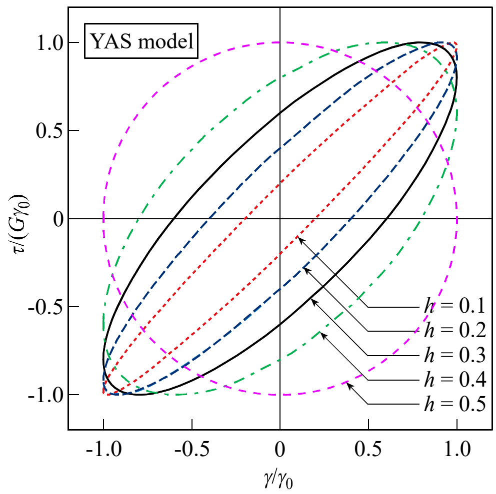

Fig. 2 Damping ratio dependency of stress–strain relationships of YAS model

$$m\ddot{\bar{u}} + C\dot{\bar{u}} + k\bar{u} = e^{i\omega t} \tag{28}$$

where $\bar{u}$ denotes displacement of the mass. The natural circular frequency and the critical damping constant β_1 of this system are $\omega_0 = \sqrt{k/m}$ and

$$2\beta_1 = \frac{C}{\sqrt{mk}}, \tag{29}$$

respectively. Since it is noted that the parameter β_1 defined in Eq. (29) is different from β defined in Eq. (5), subscript 1 is used to distinguish this from β. The steady-state solution of Eq. (28) becomes as follows.

$$\bar{u} = \frac{1}{\omega_0^2 - \omega^2 + 2i\beta_1\omega\omega_0}\frac{e^{i\omega t}}{m} = \frac{1}{1-\alpha^2 + 2i\beta_1\alpha}\frac{e^{i\omega t}}{k} \tag{30}$$

where $\alpha = \omega/\omega_0$ is a tuning ratio. The displacement response is obtained from the real part of $\bar{u}$, resulting in

$$u = \frac{\cos(\omega t - \phi)}{k\sqrt{(1-\alpha^2)^2 + (2\alpha\beta_1)^2}}, \quad \tan\phi = \frac{2\alpha\beta_1}{1-\alpha^2} \tag{31a, b}$$

where ϕ denotes the angle of phase lag. This model is called the Voigt model[3] in this paper.

Next, the same problem is solved by using the Sorokin model. The equation of motion is written as follows by replacing the G in the Sorokin complex modulus with the spring constant k.

$$\begin{aligned} m\ddot{\bar{u}} + \bar{k}^*\bar{u} &= e^{i\omega t} \\ \bar{k}^* &= k(1+2i\beta_1) \end{aligned} \tag{32a, b}$$

The steady-state solution of this equation of motion is given as

$$\bar{u} = \frac{1}{\omega_0^2 - \omega^2 + 2i\beta_1\omega_0^2}\frac{e^{i\omega t}}{m} \tag{33}$$

The displacement response is obtained by taking the real part of Eq. (33), resulting in

$$u = \frac{1}{k\sqrt{(1-\alpha^2)^2 + (2\beta_1)^2}}\cos(\omega t - \phi), \quad \tan\phi = \frac{2\beta_1}{1-\alpha^2} \tag{34a, b}$$

This solution is different from Eq. (31). Then Lysmer proposed a new complex modulus[2)].

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[3] It is noted that we do not discuss the Voigt model itself in this section, but discuss a SDOF system in which the Voigt model is used for the spring. The Lysmer model and the YAS model are used in the same way.

$$\bar{k}_L^* = k(1 - 2\beta_1^2 + 2i\beta_1\sqrt{1-\beta_1^2}) \tag{35}$$

The following displacement is obtained by solving the equation of motion using this complex modulus as

$$u = \frac{\cos(\omega t - \phi)}{k\sqrt{(1-\alpha^2)^2 + (2\alpha\beta_1)^2}}, \quad \tan\phi = \frac{2\beta_1\sqrt{1-\beta_1^2}}{1-\alpha^2-2\beta_1^2} \tag{36a, b}$$

We can see that the amplitudes are the same from the comparison of Eqs. (31a) and (36a). However, from Eqs. (31b) and (36b), the angles of phase lag are different, therefore, the load $\cos\omega t$ displacement u relationships (restoring-force characteristics) are different. The same solution is obtained by Udaka et al.[11]. They only pointed out that the phase difference becomes $2\beta_1$ rad at the maximum, but the discussion shown in the following was not made.

Equation (35) appears without any explanation in Lysmer's proposal. Kramer[12] pointed out that Eq. (35) is a choice for the displacement that equals to that of Eq. (31a). He also described that, for the usual small damping ratio considered in earthquake engineering problems, the β_1^2 terms can be neglected resulting in Eq. (32a). We try a new discussion on this equation.

At first, the complex modulus is set as $\bar{k}^* = k(k_r + ik_i)$ and substituted into Eq. (32a) to obtain the displacement. The condition that this displacement equals to that of Eq. (31a) yields

$$1 - 2\alpha^2 + 4\alpha^2\beta_1^2 = k_r^2 - 2k_r\alpha^2 + k_i^2 \tag{37}$$

This means that there are countless complex moduli that make the displacement the same as Eq. (31a). For example, we can obtain $k_i = 2\beta_1(1-2\beta_1^2)^{0.5}$ by substituting $k_r = 1-2\beta_1^2$. For another example, let set $k_i = 2\beta_1$ in order to obtain the same hysteretic absorption energy or damping ratio, we obtain

$$k_r = \alpha^2 \pm \sqrt{(1-\alpha^2)(1-\alpha^2-4\beta_1^2)} \tag{38}$$

Since the tuning ratio α remains in the real part of the complex modulus, the resulting stress–strain relationships are frequency dependent. Possibly, Eq. (35) is the only complex modulus that does not include the tuning ratio. In addition, it is clear that the complex modulus that agrees with the Voigt model in terms of both the maximum displacement and the hysteretic absorption energy is impossible by the Lysmer model.

For comparison purpose, the response of the YAS model is calculated by using the same procedure, resulting in

$$u = \frac{\cos(\omega t - \phi)}{k\left(\sqrt{(\sqrt{1-4\beta_1^2} - \alpha^2)^2 + (2\beta_1)^2}\right)}, \quad \tan\phi = \frac{2\beta_1}{\sqrt{1-4\beta_1^2} - \alpha^2} \tag{39a, b}$$

Here, as in the other models, G and h are replaced by k and β_1.

The displacement time history and the restoring force characteristics of the four models are compared in Fig. 3. Here, displacement is expressed as uk (Modified displacement) by multiplying with the spring constant k. As discussed in the preceding section, the displacements of the Lysmer model and the Voigt model are the same, but since their angles of phase lag are different from each other, their restoring force characteristics are different. In this sense, Lysmer's suggestion that the displacement is to be the same as that of the Voigt model does not seem to be an important issue. In addition, the discussion by Lysmer is made on a SDOF system, and there is no explanation of how this relates to the shear behavior for the seismic response of a ground. This is discussed in detail in section 6.1, and the characteristics and problems of the SDOF system are discussed in section 5.2.

4.2 Mechanical characteristics of Lysmer model

The complex spring k^* proposed by Lysmer is shown in Eq. (35), and β_1 is called a fraction of critical damping which is defined in Eq. (29).

After the discussion in the previous section, Lysmer extended the discussion into the multiple-degree-of-freedom system by a finite element method. He described "in order to get good agreement between a modal analysis and a complex response with the complex moduli one must choose the following relationship (Eq. (40)) between the real modulus G which is used in the modal analysis and the complex modulus G^* used in the complex response analysis"

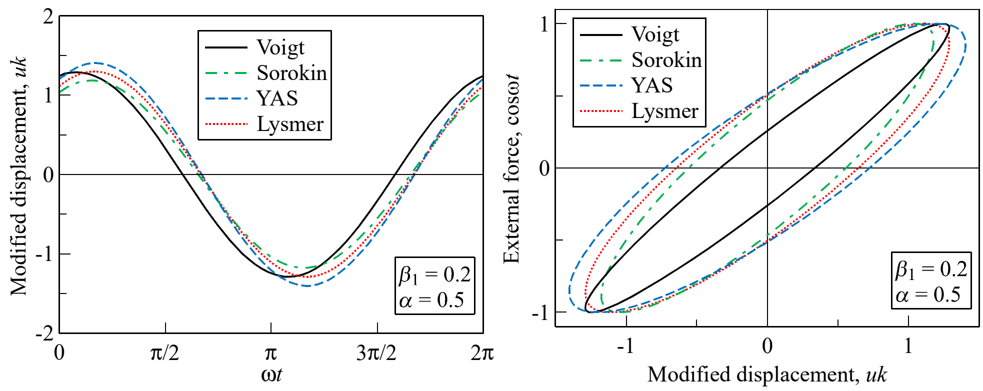
$$G_L^* = G(1-2\beta_1^2 + 2i\beta_1\sqrt{1-\beta_1^2}) \tag{40}$$

where G is the stiffness obtained in the laboratory test. He also described that the Sorokin model can be used when the damping ratio is small, and he showed the complex moduli for S- and P-wave velocities. However, he replaced β_1 in Eq. (5) with β in Eq. (5), i.e.,

$$G_L^* = G(1-2\beta^2 + 2i\beta\sqrt{1-\beta^2}) \tag{41}$$

without explanation. Since the complex modulus is used in the analysis of the ground, we call this model the Lysmer model. The complex stress $\overline{\tau}$ against the complex strain $\overline{\gamma}$ (complex stress–strain relationships) and the stress–strain relationships as the real part of the complex stress–strain relationships become as follows.

$$\begin{aligned}
&\overline{\tau} = \overline{G}_L^*\overline{\gamma} = G\gamma_0(1-2\beta^2 + 2i\beta\sqrt{1-\beta^2})(\cos\omega t + i\sin\omega t)\\
&\tau = G\left\{(1-2\beta^2)\gamma \pm 2\beta\sqrt{1-\beta^2}\sqrt{\gamma_0^2-\gamma^2}\right\} = G\gamma_0\cos(\omega t+\phi) \qquad (42a, b, c)\\
&\tan\phi = \frac{2\beta\sqrt{1-\beta^2}}{1-2\beta^2}
\end{aligned}$$



(a) Displacement time history (b) Load–displacement relationships

Fig. 3 Comparison of the behavior of SDOF system

Although it is not written in Lysmer's proposal[2)], the maximum stress of the Lysmer model is $G\gamma_0$, which is the same maximum stress in the laboratory test. This is a very important mechanical characteristic of the Lysmer model.

The hysteretic absorption energy ΔW_L is calculated from Eq. (42b) as

$$\Delta W_L = 2\beta G\gamma_0^2 \pi\sqrt{1-\beta^2} \tag{43}$$

The relationships between the material damping constant β and the damping ratio h is obtained by substituting ΔW_L into Eq. (1a) as

$$h = \frac{1}{4\pi}\frac{2\beta G\gamma_0^2\pi\sqrt{1-\beta^2}}{G\gamma_0^2/2} = \beta\sqrt{1-\beta^2} \tag{44}$$

The material damping constant β is obtained as

$$\beta = \sqrt{\frac{1-\sqrt{1-4h^2}}{2}} \tag{45}$$

The relationships between β and h are shown in Fig. 4. The term in the root on the right-hand side becomes negative when $h > 0.5$. Therefore, the conversion from the damping ratio to the material damping constant is possible only when $h \le 0.5$ ($\beta \le 1/\sqrt{2}$). In the current practice, the material damping constant β is used instead of the damping ratio h, but it results in smaller hysteretic absorption energy.

By substitution of Eq. (41) into Eq. (45) in order to express the Lysmer model as a function of the damping ratio h, we obtain

$$\overline{G}_L^* = G(1-2\beta^2 + 2i\beta\sqrt{1-\beta^2}) = G(\sqrt{1-4h^2} + 2ih) \tag{46}$$

The right-hand side of this equation is the same as the YAS model; the Lysmer model yields the YAS model by using the conversion in Eq. (45).

5. DISCUSSION OF APPLICABILITY OF LYSMER MODEL

We discuss the applicability of the Lysmer model from two points of view.

5.1 Applicability of Lysmer model

When Christian et al. introduced the Lysmer model[13)], they pointed out the following. Since the difference between the Sorokin model and the Lysmer model are small when $\beta < 0.3$, the use of the Sorokin model is suggested. They also raised a question for the equivalent linear method using a complex modulus for β greater than 0.3 because of the lack of test data. Moreover, the imaginary part of the Sorokin model monotonically increases with β, but the Lysmer model has a problem that the real part can become negative and $\overline{G}^* = -G$ when $\beta = 1$. Considering these points, they felt that the Sorokin model is enough for practical use.

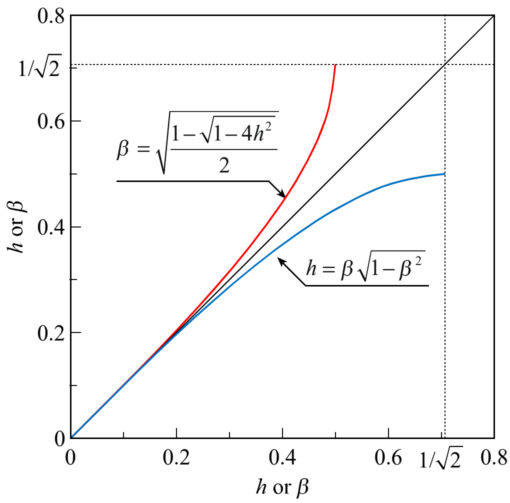


Fig. 4 Relationships between material damping constant and damping ratio in Lysmer model

When one of the authors collected hundreds of cyclic shear test data[14)], almost all maximum damping ratios satisfied $h < 0.3$. Therefore, the suggestion by Christian et al. seems rational. However, they did not discuss the amount of error; they just described that the error is small. This is discussed in the next chapter.

Another issue that Christian et al. pointed out is the behavior of the Lysmer model at large β. They just pointed out an inconvenient point when $\beta = 1$. This is discussed in detail here.

The relationships of the real and imaginary parts of both the Sorokin and the Lysmer models and the material damping constant β are shown in Fig. 5. The real part (normalized by G in the figure) is constant and the imaginary part is a straight line that increases with β. On the other hand, in the case of the Lysmer model, the real part decreases with β and becomes zero at $\beta = 1/\sqrt{2}$. The imaginary part is close to that of the Sorokin model when β is small, but it separates from that of the Sorokin model as β increases; the imaginary part has an extreme value 1 at $\beta = 1/\sqrt{2}$ and becomes zero at $\beta = 1$.

The stress (normalized by $G\gamma_0$)–strain (normalized by γ_0) relationships are shown in Fig. 6 to see the effect of β. The stress–strain relationships are elliptic in shape when β is small, and transforms to a circle as β increases. It becomes a complete circle at $\beta = 1/\sqrt{2}$ because the real part becomes zero. When β increases further, they again become elliptic in shape, but the slope angle becomes negative when $\beta > 1/\sqrt{2}$, whereas it is positive when $\beta < 1/\sqrt{2}$. They become a straight line at $\beta = 1$. Thus we can conclude that the applicable range of the Lysmer model is $\beta \le 1/\sqrt{2}$.

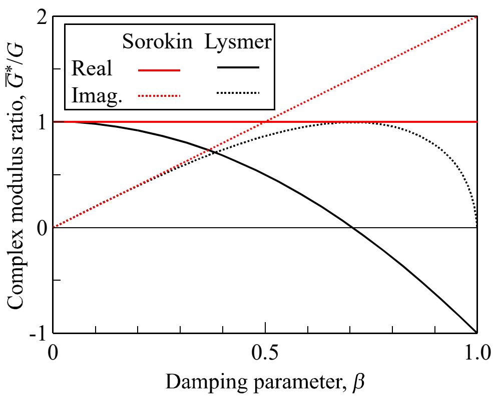


Fig. 5 Material damping constant dependency of complex moduli

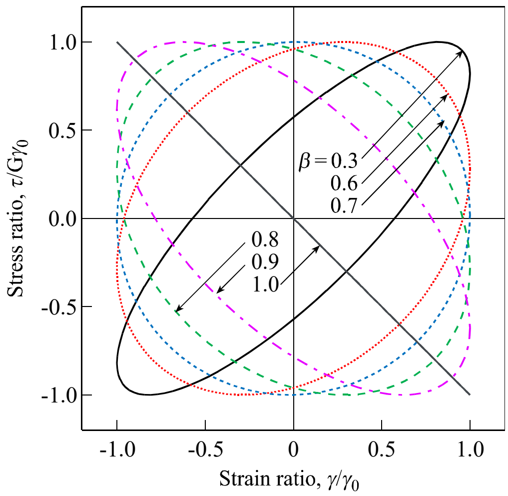


Fig. 6 Stress–strain curves of Lysmer model

On the other hand, Kramer[12] pointed out, as described in section 4.1, that the difference of the angle of phase lag can be approximated by $2\beta/(1+\alpha)$ when β is small so that β^2 is negligible. He also described that the motion with viscous damping can be expressed by a non-damped system (velocity proportional term is not included in Eq. (32a)) using the complex modulus. He also mentioned that this can be done only for the harmonic vibration problem. Although the discussion by Kramer is qualitative and no quantitative discussion was made, he seems to consider that the Sorokin model is sufficient.

5.2 Vibration problem under harmonic external load

It is usual to use a damped free vibration system in the discussion of the damping ratio and the critical damping ratio. On the other hand, the amplification ratio is usually of interest when dealing with harmonic excitation. However, we change our focus and discuss the relationships between the restoring force characteristics and the damping ratio.

The real part of the equation of motion of the SDOF system, shown in Eq. (28), is

$$m\ddot{u} + C\dot{u} + ku = \cos\omega t \tag{47}$$

The restoring force of this system Q is obtained from Eq. (47) by using Eq. (29) and the relation $\omega_0 = \sqrt{k/m}$ as

$$Q = -m\ddot{u} + \cos\omega t = C\dot{u} + ku = \frac{2\beta_1 k}{\omega_0}\dot{u} + ku \tag{48}$$

The circular frequency of the displacement equals to that of the excitation under the steady-state.

$$u = u_0\cos(\omega t - \phi) \tag{49}$$

where ϕ denotes an angle of phase lag from the external load. Substituting this equation into Eq. (48) and, using $\sin(\omega t - \phi) = \pm\sqrt{1-\cos^2(\omega t - \phi)}$, we obtain

$$Q = ku \mp 2\beta_1 k\frac{\omega}{\omega_0}\sqrt{u_0^2 - u^2} \tag{50}$$

The hysteretic absorption energy and strain energy are calculated from the Q–u relationships as

$$\begin{aligned} \Delta W &= \int_{-u_0}^{u_0}\left\{\left(ku + 2\beta_1 k\frac{\omega}{\omega_0}\sqrt{u_0^2-u^2}\right) - \left(ku - 2\beta_1 k\frac{\omega}{\omega_0}\sqrt{u_0^2-u^2}\right)\right\}du = 2k\beta_1\frac{\omega}{\omega_0}\pi u_0^2 \\ W &= \frac{1}{2}ku_0^2 \end{aligned} \tag{51}$$

Therefore, the damping ratio h yields

$$h = \frac{1}{4\pi}\frac{\Delta W}{W} = \beta_1\frac{\omega}{\omega_0} \tag{52}$$

This equation indicates that $\beta_1 = h$ only when $\omega = \omega_0$, i.e., the circular frequency of the excitation equals the natural circular frequency of the system, which holds in the steady-state vibration under the harmonic excitation. Thus the attempt by Lysmer, the comparison between the viscous damping and the complex modulus of the SDOF system, is not considered to be mechanically significant.

The difference between the vibration problem and the analysis of the ground is discussed in section 6.1.

6. DISCUSSION

6.1 Vibration theory and complex modulus for analysis of ground

In many books or technical papers, β_1 defined in Eq. (29) is called the critical damping ratio or the fraction of critical damping and is denoted as the damping ratio *h* instead of β_1. This comes because the damping coefficient corresponding to β_1 is a ratio of the viscous coefficient $C = 2\sqrt{mk}$ under which the system does not vibrate in the free vibration problem of the SDOF system. On the other hand, in the SDOF system with nonlinear characteristics, the hysteretic absorption energy is considered to express damping and it has been tried to express this by using β_1. When β_1 is sufficiently small, $h = \beta_1$ by using Eq. (1a) (e.g., Tajimi[15)]). Therefore, as explained in footnote 1, *h* and β_1 have not been differentiated, and have been used in the same meaning.

However, in the analysis of grounds, the definition of β_1 in Eq. (29) does not make sense because it includes mass *m*; *h* defined in Eq. (1a) is used as the output of the cyclic shear test. This *h* is called the damping constant, damping ratio[16)], hysteretic damping rate[17)], etc. It should be converted to the stress–strain relationships to conduct the seismic response analysis.

The material damping constant β defined in Eq. (5) is called the "critical damping ratio" in SHAKE[1)]. It seems that they adopted the terminology the terminology from the vibration theory of the SDOF system because SHAKE uses the Sorokin model in which $\beta = h$. However, the β used in SHAKE as defined in Eq. (5) is different from the critical damping ratio in the SODF system, Eq. (29), but there is no explanation in SHAKE[1)].

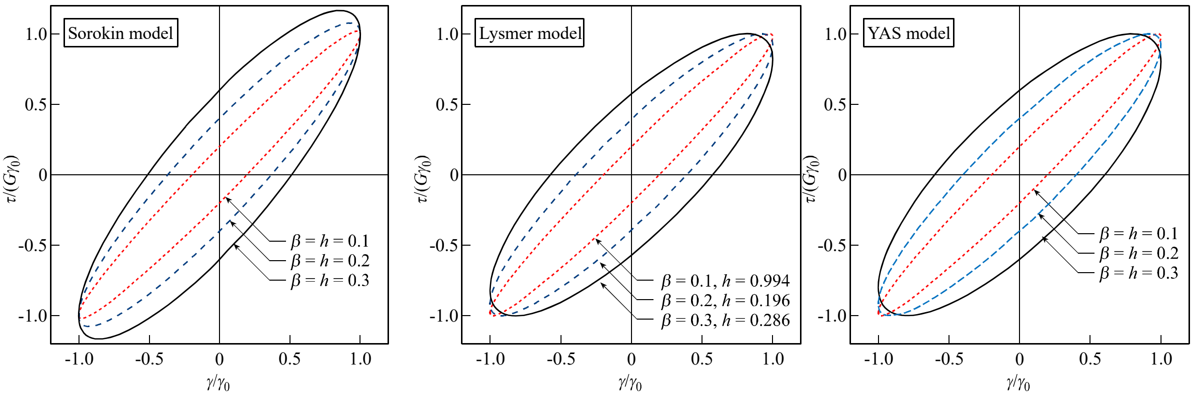
Lysmer[2)] called β_1 the fraction of critical damping, which also seems to be the same idea with SHAKE. However, as explained in section 4.2, $\beta \neq h$. Therefore, the terminology is more impossible than SHAKE.

We introduced the parameter β in Eq. (5) by focusing on the stress–strain relationships that do not have frequency dependence. This definition is quite different from the definition in Eq. (29). We cannot find the relevant technical terms on this parameter in past research. As mentioned above, it is called a critical damping ratio, but this does not seem to be a relevant term. Thus we named it a material damping constant. Definition of the complex modulus for the seismic ground response analysis is possible apart from the discussion of the vibration of a SDOF system by using this parameter to the complex modulus to express stress–strain relationships of soil.

Since the result of the laboratory test is expressed as the damping ratio *h*, it is better to use *h* to express the stress–strain relationships. The Sorokin model and the YAS model satisfy this requirement because $\beta = h$. On the other hand, since $\beta \neq h$ in the Lysmer model, β cannot be used instead of *h*. However, it becomes possible to use *h* by using the conversion in Eq. (45), in which case the Lysmer model becomes the same as the YAS model as shown in 4.2. If β is used instead of *h* in the Lysmer model, the damping ratio or the hysteretic absorption energy is about 5 % underestimated when $\beta = 0.3$. This means that the error of the hysteretic absorption energy of the Lysmer model is usually less than 5 %. If this error is acceptable, the use of the Lysmer model is possible, but in order to eliminate the error, β should be converted to *h* by using Eq. (45) in the Lysmer model.

6.2 Comparison of complex moduli

Three complex moduli, the Sorokin model $\overline{G}_S^*$, the Lysmer model $\overline{G}_L^*$, and the YAS model $\overline{G}_Y^*$, are compared in this section.



(a) Sorokin model (b) Lysmer model (c) YAS model

Fig. 7 Comparison of stress–strain relationships

Stress–strain relationships of the three complex moduli are compared in Fig. 7 with the material damping constants 0.1, 0.2, and 0.3. Here, the vertical axes are normalized by γ_0 and the horizontal axes are normalized by $G\gamma_0$.

As pointed out in chapter 3, the maximum shear stress ratio (ratio of the maximum shear stress to the maximum shear stress $G\gamma_0$ of the cyclic shear stress result) is greater than 1 in the Sorokin model. On the other hand, those in the Lysmer model and the YAS model are just 1.

The hysteresis curve of the Lysmer model (Fig. 7(b)) looks thinner than that of the YAS model (Fig. 7(c)) because, as shown in Eq. (44), h is smaller than β in the Lysmer model. In order to see the difference, the hysteretic curves are compared in Fig. 8 at the same β and h values in the Lysmer model. The difference is ridiculously small when the values are 0.1 and 0.2. A slight difference is seen when $\beta = 0.3$; as shown in the preceding section, h is about 5% smaller when $\beta = 0.3$.

The real part and the imaginary part of the complex moduli are shown in Fig. 9 against the damping ratio or the material damping constant. Here, the vertical axes are normalized by the secant modulus G. The horizontal axes are shown up to 0.8 to see the characteristics of the model well although the practically important damping ratio is less than about 0.3. It is also noted that the curves of the Lysmer model are the same as those shown in Fig. 5. The difference between the Lysmer model and the YAS model is small in the practically important range of the damping ratio ($h \le 0.3$), but it becomes large as the damping ratio becomes larger than 0.4.

As shown in Fig. 9(a), the real part is constant in the Sorokin model, but the real parts become smaller as h or β in the other models. In addition, the real parts become zero when $h = 0.5$ in the YAS model and $\beta = 1/\sqrt{2}$ in the Lysmer model. The hysteresis curve becomes a circle, and this is the applicable range of the model. The applicable range becomes the same for both models when using the conversion in Eq. (44) or Eq. (45).

The imaginary part of the Sorokin model and YAS model are the same, therefore the damping ratio of these models is the same because, as can be seen in Eq. (14) or Eq. (22), the hysteretic absorption energy is determined by the imaginary part of the complex modulus. On the other hand, the imaginary part is smaller in the Lysmer model than that of the other two models, which indicates that the hysteretic absorption of the Lysmer model is smaller than that of the other two models when the material damping constant β is the same. However, as can be seen in Fig. 7, there is no significant difference in the hysteretic curves.

7. CONCLUSION

A new complex modulus, named YSA (Yoshida–Adachi–Sorokin) model, for the cyclic stress–strain relationships of the soil, gives the same maximum stress and the damping ratio as the ones by the cyclic shear test. Then three complex moduli, the Sorokin model used in the original SHAKE[1], the Lysmer

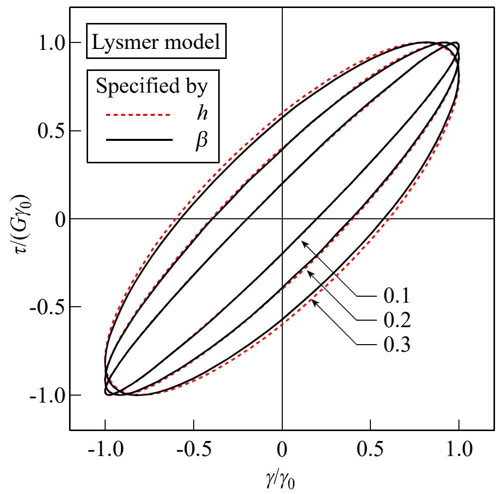
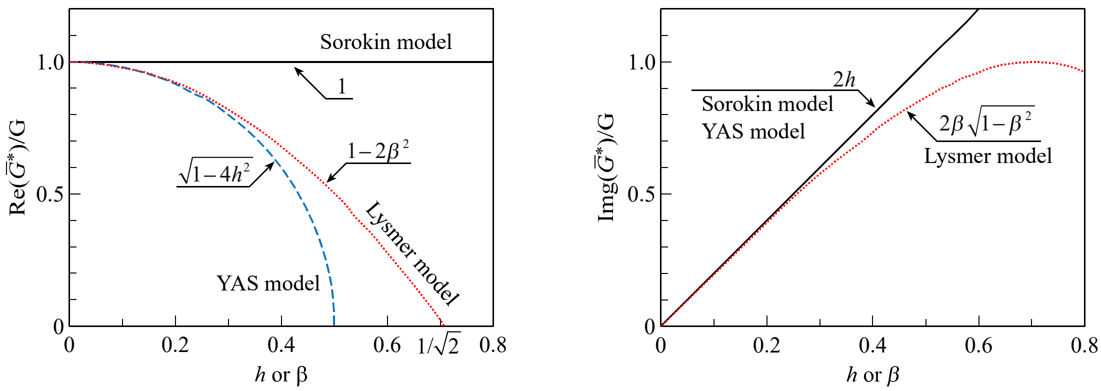


Fig. 8 Comparison of hysteretic curve when h and β are same value



(a) Real part of complex modulus (b) Imaginary part of complex modulus

Fig. 9 Damping ratio h and material damping constant β dependency of complex moduli

model[2)] proposed to improve the Sorokin model, and the YAS model proposed in this paper are compared and discussed. Finally, we obtained the following conclusions.

(1) A new parameter β named the material damping constant is defined. This parameter is different from the damping ratio (or the critical damping ratio which is defined as β_1 in this paper) used in the analysis of a SDOF system. The former is the characteristic value of the material and the latter is the characteristic value of the system.

(2) The Sorokin model overestimates the maximum stress by $\sqrt{1+4h^2}$ times larger than the one in the laboratory test.

(3) The YAS model is applicable within the damping ratio h of 0.5 or less. This means that the complex modulus whereby both the maximum stress and the hysteretic absorption energy agree with those by the laboratory test is possible only when $h \le 0.5$.

(4) The complex spring constant proposed by Lysmer was proposed to get the same displacement amplitude with a SDOF system with viscous damping under harmonic loading. However, although the displacement is the same, the restoring-force characteristics are different. In addition, he replaced the critical damping ratio with the material damping constant to use in the analysis of the ground, but the mechanical background was not shown.

(5) The maximum stress of the Lysmer model is the same as the one in the laboratory test.

(6) The material damping constant β is used instead of the damping ratio h in the Lysmer model, which underestimates the hysteretic absorption energy because the damping ratio calculated by using β instead of h is smaller than h.
(7) An equation is derived to obtain the damping ratio h from the material damping constant β so that the hysteretic absorption energy is the same as the one in the laboratory test. By using this equation, the Lysmer model yields the YAS model.
(8) The difference in the hysteretic absorption energy is small between the Lysmer model and the YAS model within the practically important range of the damping ratio $h \leq 0.3$. The error is about 5 % even at $h = 0.3$. If this difference is not a problem, the use of the Lysmer model may be possible, but the use of the YAS model is suggested because the mechanism is clear and both the maximum stress and hysteretic absorption energy in the YAS model agree with those of the test.

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